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Best Interpolation in a Strip

A. L. DONTCHEV*

Mathematical Reviews, 416 Fourth Street, Ann Arbor, Michigan 48107, U.S.A.

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Given points $0 = t_1 < \cdots < t_n = 1$, real numbers y_i, $i = 1, ..., n$, and piecewise linear splines e and d with knots t_i such that $e(t_i) < y_i < d(t_i)$ we consider the problem to find a function f with minimal L_2-norm of the second derivative and which satisfies the conditions $f(t_i) = y_i$, $i = 1, ..., n$, and $e(t) \leqslant f(t) \leqslant d(t)$ for all $t \in [0, 1]$. We prove that the solution of this problem is a cubic spline which depends continuously on the data. © 1993 Academic Press, Inc.

1. INTRODUCTION

Given points $0 = t_1 < \cdots < t_n = 1$, and real numbers e_i, y_i, d_i, $i = 1, ..., n$, let e and d be continuous functions on $[0, 1]$ that are linear between t_i and t_{i+1} and $e(t_i) = e_i$, $d(t_i) = d_i$ for $i = 1, ..., n$. Let $W^{2,2}$ be the Sobolev space of real-valued functions over $[0, 1]$ with square-integrable second derivatives, equipped with the norm

$$\|f\|_{W^{2,2}} = |f(0)| + |f'(0)| + \|f''\|,$$

where $\|\cdot\|$ is the L^2 norm. Define the set G as

$$\{f \in W^{2,2}: f(t_i) = y_i,\ i = 1, ..., n, \text{ and } e(t) \leqslant f(t) \leqslant d(t) \text{ for all } t \in [0, 1]\}.$$

In this paper we consider the problem

$$\text{minimize } \|f''\| \text{ subject to } f \in G. \tag{1}$$

We prove that if the data (e_i, y_i, d_i), $i = 1, ..., n$, are in the set

$$D = \{\delta \in R^{3n}: \delta_i = (e_i, y_i, d_i),\ e_i < y_i < d_i,\ i = 1, ..., n\}$$

then the problem (1) has a unique solution, and moreover, the solution is a cubic spline with no more than four additional knots in each interval

* On leave from the Institute of Mathematics, Bulgarian Academy of Sciences.

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$[t_i, t_{i+1}]$ such that the upper or the lower constraint is active between the new knots. We also show that the solution is locally 1/2-Lipschitz with respect to the data, that is, for every $\delta_0 \in D$ there exist a constant $c > 0$ and a neighbourhood $\mathcal{N}$ of δ_0 such that, for every $\delta^1, \delta^2 \in \mathcal{N}$ the corresponding solutions f^1, f^2 satisfy

$$\|f^1 - f^2\|_{W^{2,2}} \leqslant c\,|\delta^1 - \delta^2|^{1/2}.$$

This means that, in a sense, problem (1) is well posed for data in D.

Problem (1) is a generalization of the classical spline-interpolation problem and, especially, of the best positive interpolation considered by Opfer and Oberle [8], where $e = 0$ and $d = +\infty$. Related convex and monotone best interpolation problems are studied in [2–4, 7].

Our main result follows:

THEOREM. *For every datum in the set D the solution f^* of the problem (1) is a natural cubic spline with knots $\{t_i\}$ and no more than 4n additional knots, and satisfies the conditions*:

(1) $f''(0) = f''(1) = 0$.

(2) *Every interval $[t_i, t_{i+1}]$ contains no more than four new knots $t_i < \tau_1 \leqslant \tau_2 < \sigma_1 \leqslant \sigma_2 < t_{i+1}$ (or $t_i < \sigma_1 \leqslant \sigma_2 < \tau_1 \leqslant \tau_2 < t_{i+1}$) such that $f(t) = d(t)$ on $[\tau_1, \tau_2]$ and $f(t) = e(t)$ on $[\sigma_1, \sigma_2]$.*

(3) *Let τ be an additional knot. If $f(\tau) = e(\tau)$ then $f'''(\tau - 0) \leqslant f'''(\tau + 0)$ and if $f(\tau) = d(\tau)$ then $f'''(\tau + 0) \leqslant f'''(\tau - 0)$.*

Moreover, f^* is a locally 1/2-Lipschitz function of the data in the set D.

The proof of the existence and uniqueness is standard and is given in Section 2. In Section 3 we rewrite problem (1) as an optimal control problem and present a lemma on Lagrange duality. Then in Section 4 we show that the solution is a cubic spline and in Section 5 we complete the proof of the theorem. In Section 6 we discuss a general constrained best interpolation problem.

2. Existence and Uniqueness

Fix $\{e_i, y_i, d_i\}_{i=1}^n$ in D. Let f_0 be the piecewise linear spline, interpolating $\{t_i, y_i\}$. Given an integer m, in every interval $[t_i, t_{i+1}]$ we introduce new knots $\tilde{t}_j^i = t_i + j(t_{i+1} - t_i)/m$, $j = 0, 1, ..., m-1$. Let S_m be the cubic spline that interpolates f_0 at $\tilde{t}_j^i$. It is known (see Sharma and Meier [10]) that, since f_0 is Lipschitz, S_m converges uniformly in [0, 1] to f_0 as $m \to \infty$ (actually, from more recent results it follows that $\max_{t\in[0,1]} |f_0(t) - S_m(t)| = O(1/m^2)$). Then

$$S_m(t_i) = y_i,\ i = 1, ..., n, \quad \text{and} \quad e(t) < S_m(t) < d(t) \quad \text{for all } t \in [0, 1], \tag{2}$$

for sufficiently large m. In particular, the set $\{g \in L_2 : g = f'' \text{ for } f \in G\}$ is nonempty. Since it is weakly closed and convex, we conclude that the problem (1) has a solution and that the solution is unique.

3. A Duality Lemma

To show that the solution is a spline we first rewrite (1) as an optimal control problem, denoting $f = x_1$, $f' = x_2$, $f'' = u$: Find an absolutely continuous function $x = (x_1, x_2)$ and an $u \in L_2$ which

$$\begin{gathered} \text{minimize } \|u\|^2 \text{ subject to} \\ x_1' = x_2,\ x_2' = u,\ x_1(t_i) = y_i,\ i = 1, ..., n, \\ e(t) \leqslant x_1(t) \leqslant d(t) \quad \text{for all} \quad t \in [0, 1]. \end{gathered} \tag{3}$$

This is a convex and smooth minimum problem. Below we derive optimality conditions for this problem, using the general scheme from the theory of extremal problems, based on the theorem for separation of convex sets (see [5]).

Lemma. *Let $x^* = (x_1^*, x_2^*)$ and u^* be the solution of (3). Then u^* is an absolutely continuous function with $u^*(0) = u^*(1) = 0$ and there exist real numbers l_i, $i = 1, ..., n$, and nonnegative regular measures μ_1 and μ_2, supported on the sets $T_1 = \{t \in [0, 1]: x_1^*(t) = e(t)\}$ and $T_2 = \{t \in [0, 1]: x_1^*(t) = d(t)\}$, respectively, such that*

$$(u^*)'(t) = l_i + \int_t^1 d(\mu_1 - \mu_2) \qquad \text{for a.e.} \quad t_i \leqslant t \leqslant 1,\ i = 1, ..., n. \tag{4}$$

Proof of the Lemma. Let $\hat{c} = \|u^*\|^2$. Consider the following subsets in $R \times C(R^2) \times R^n \times C(R^2)$:

$$A = \{(a, b, c, h) : a \leqslant \hat{c},\ b = 0,\ c = 0,\ h(t) \leqslant 0 \text{ for all } t \in [0, 1]\},$$

$$B = \{(a, b, c, h) : \text{there exist } x \in C(R^2),\ u \in L_2,\ \alpha \in R^2 \text{ such that}$$

$$\|u\|^2 \leqslant a,\ x_1(t) - \alpha_1 - \int_0^t x_2\, ds = b_1(t),\ x_2(t) - \alpha_2 - \int_0^t u\, ds = b_2(t),$$

$$c_i = x_1(t_i) - y_i,\ i = 1, ..., n,\ e(t) - x_1(t) \leqslant h_1(t),$$

$$x_1(t) - d(t) \leqslant h_2(t) \text{ for all } t \in [0, 1]\}.$$

From the existence of the spline S_m, satisfying (2), we see that the set B has nonempty interior and, from the optimality, it follows that this interior does not meet A. Hence the sets A and B can be separated by a hyperplane. Then there exist $r \in R$, a regular Borel measure $\nu = (\nu_1, \nu_2)$, a vector $\lambda \in R^n$, and a regular Borel measure $\mu = (\mu_1, \mu_2)$, not all zero and such that

$$ra^1 + \int_0^1 b^1\, d\nu(t) + \lambda^T c^1 + \int_0^1 h^1\, d\mu(t)$$
$$\geqslant ra^2 + \int_0^1 b^2\, d\nu(t) + \lambda^T c^2 + \int_0^1 h^2\, d\nu(t), \tag{5}$$

whenever $(a^1, b^1, c^1, h^1) \in B$ and $(a^2, b^2, c^2, h^2) \in A$, where the superscript ^{T} denotes transposition. By choosing particular points in A and B we find the properties of r, ν, λ, μ:

(a) $r \geqslant 0$. Take $a^2 = \hat{c} - 1$, $a^2 = \|S_m''\|^2$, $b^1 = b^2 = 0$, $c^1 = c^2 = 0$, $h^1 = h^2 = 0$.

(b) *μ is nonnegative.* Take $a^1 = a^2 = \hat{c}$, $b^1 = b^2 = 0$, $c^1 = c^2 = 0$, $h^1 = 0$. Then, taking into account (2), from (5) we obtain $0 \geqslant \int h^2\, d\mu$ for every nonpositive continuous function h^2.

(c) *μ_i is supported on the set T_i, $i = 1, 2$.* Take $a^1 = a^2 = \hat{c}$, $b^1 = b^2 = 0$, $c^1 = c^2 = 0$, $h_1^1 = e - x_1^*$, $h_2^1 = x_1^* - d$, and $h^2 = 0$. Then from (5) and (b) we obtain

$$\int_0^1 (e - x_1^*)\, d\mu_1 = 0 \qquad \text{and} \qquad \int_0^1 (x_1^* - d)\, d\mu_2 = 0.$$

We introduce the Lagrange functional

$$\mathscr{L}(x, \alpha, u) = r\, \|u\|^2 + \int_0^1 \left[x_1(t) - \alpha_1 - \int_0^t x_2\, ds \right] d\nu_1$$
$$+ \int_0^1 \left[x_2(t) - \alpha_2 - \int_0^t u\, ds \right] d\nu_2 + \sum_{i=1}^n \lambda_i (x_1(t_i) - y_i)$$
$$+ \int_0^1 (e - x_1)\, d\mu_1 + \int_0^1 (x_1 - d)\, d\mu_2.$$

Then (5) implies that

$$\mathscr{L}(x^*, x^*(0), u^*) \leqslant \mathscr{L}(x, \alpha, u) \tag{6}$$

for every $x \in C(R^2)$, $\alpha \in R^2$, and $u \in L_2$.

If we assume that $r=0$ and substitute $x_1=S_m$, $\alpha_1=S_m(0)=y_1$, $x_2=S'_m$, $\alpha_2=S'_m(0)$, $u=S''_m$ in (6), where S_m satisfies (2), we obtain

$$0\leqslant\int_0^1 (e-S_m)\,d\mu_1+\int_0^1 (S_m-d)\,d\mu_2\leqslant 0;$$

hence μ_1 and μ_2 are zero.

Let P_n be a polynomial with values $P_n(t_i)=y_i-\lambda_i$, $i=1, ..., n$. Let $x^1=P_n$, $\alpha_1=P_n(0)$, $x_2=P'_n$, $\alpha_2=P'_n(0)$, and $u=P''_n$. Then from (6) we get

$$0\leqslant-\sum_{i=1}^{n}\lambda_i^2,\qquad \text{that is,}\quad \lambda=0.$$

Finally, given a continuous function δ, substituting $x_1=\delta$, $\alpha_1=\alpha_2=0$, $x_2=0$, $u=0$, in the right-hand side of (5) we obtain

$$0\leqslant\int_0^1 \delta\,dv_1;$$

hence $v_1=0$. Similarly, $v_2=0$. We obtain that all Lagrange multipliers are zero, which is a contradiction. Hence, $r>0$ and then one can assume that $r=0.5$.

Since the Lagrange functional is convex and Fréchet differentiable in $C(R^2)\times R^2\times L_2$, $(x^*, x^*(0), u^*)$ will be its stationary point. Define

$$p_1=\int_t^1 dv_1\qquad\text{and}\qquad p_2=\int_t^1 dv_2.$$

Since the measures v_1 and v_2 are regular, the functions p_1 and p_2 are of bounded variation and are continuous from the left. Differentiating with respect to α we obtain that $p_1(0)=p_2(0)=0$. Exchanging the order of integration gives

$$0.5\int_0^1 u(t)^2\,dt-\int_0^t\left[\int_0^t u(s)\,ds\right]dv_2=0.5\int_0^1 u(t)^2\,dt-\int_0^1 u(t)\,p_2(t)\,dt.$$

Thus, $\partial\mathscr{L}/\partial u=0$ yields

$$u^*=p_2.$$

Furthermore

$$\int_0^1 x_2\,dv_2-\int_0^1\left[\int_0^t x_2\,ds\right]dv_1=\int_0^1 x_2\,d\left(-p_2+\int_s^1 p_1\right).$$

Hence

$$p_2' = -p_1, \qquad p_2(0) = 0.$$

Finally, we have

$$\int_0^1 x_1\, dv_1 + \sum_{i=1}^n \lambda_i x_1(t_i) + \int_0^1 x_1\, d(\mu_2 - \mu_1)$$
$$= \int_0^1 x_1\, d\left[v_1 + \sum_{i=1}^n \lambda_i (t - t_i)'_+ + \mu_2 - \mu_1\right]$$

and therefore

$$p_1(t) = -\int_t^1 d\left[\sum_{i=1}^n \lambda_i (t - t_i)'_+ + \mu_2 - \mu_1\right].$$

Denoting $\sum_{i=j+1}^n \lambda_j = l_i$ we complete the proof of the lemma.

4. The Solution Is a Cubic Spline

From (4) we obtain that the third derivative of x_1^* is constant in the set $\{t \in (t_i, t_{i+1})\colon e(t) < x_1^*(t) < d(t)\}$. This means that, within every interval $[t_i, t_{i+1}]$, in the interior of the strip x_1^* is a cubic polynomial.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1})$ for some i such that $x_1^*(\tau_1) = d(\tau_1)$ and $x_1^*(\tau_2) = d(\tau_2)$. Then $x_2^*(\tau_1) = x_2^*(\tau_2) = d'(\tau_1) = d'(\tau_2)$. Let

$$\omega(t) = \begin{cases} 0 & \text{if } t \in (\tau_1, \tau_2), \\ u^*(t) & \text{otherwise.} \end{cases} \tag{7}$$

Let $\eta_1(t) = d(t)$ and $\eta_2(t) = d'(t)$ for $t \in (\tau_1, \tau_2)$, and $\eta_i = x_i^*$, $i = 1, 2$, outside (τ_1, τ_2). Then ω, η_1, and η_2 satisfy the constraints of (3). From (7) it follows that $\|w\| \leqslant \|u^*\|$. This means that $u^*(t) = 0$ on the interval $[\tau_1, \tau_2]$. Hence, u^* is piecewise linear.

We obtain that, between every t_i and t_{i+1}, each boundary of the strip can be active only on one interval (or at one point). The ends of every such interval will be new knots, and the function will lie on the boundary between these two knots (or will touch the boundary at the new knot). Since the measures μ_1 and μ_2 are positive we obtain that the derivative of u^* will have a jump up at every "contact" point with the boundary e and will have a jump down at every contact point with d.

5. Proof of 1/2-Lipschitz Continuity

Fix some $\delta_0 \in D$. We show first that there exists $r_0 > 0$ such that the factor r in the Lagrange functional is greater than r_0 for all δ near δ_0.

Note that, without loss of generality, one may assume that

$$r + \left| \int_0^1 dv_1 \right|^2 + \left| \int_0^1 dv_2 \right|^2 + \sum_{i=1}^n \lambda_i^2 + \int_0^1 d(\mu_1 + \mu_2) = 1. \tag{8}$$

Furthermore, by (6) we see that $\|u^*\|^2$ is bounded in a neighbourhood of δ_0.

It is a standard observation that the spline S_m satisfying (2) depends continuously on the interpolation values $\{y_i\}_{i=1}^n$ as a function from R^n to $W^{2,2}$. Let ε be sufficiently small such that $e_i + \varepsilon < y_i - \varepsilon\lambda_i < d(t) - \varepsilon$ for all i and for all δ near δ_0. For every such δ and for sufficiently large m we can construct as in (2) a cubic spline S_m with the following properties: $S_m(t_i) = y_i - \varepsilon\lambda_i$, $i = 1, ..., n$, and $e(t) + \varepsilon < S_m(t) < d(t) - \varepsilon$ for all $t \in [0, 1]$. In (6) take $x_1 = S_m$, $x_2 = S'_m$, $u = S''_m$, $\alpha_1 = S_m(0) - \varepsilon \int_0^1 dv_1$, $\alpha_2 = S_m(0) - \varepsilon \int_0^1 dv_2$. Then, using (8), we obtain

$$\begin{aligned} r \|u^*\|^2 &\leqslant r \|S_i''\| - \varepsilon \left(\left| \int_0^1 dv_1 \right|^2 + \left| \int_0^1 dv_2 \right|^2 + \sum_{i=1}^n \lambda_i^2 + \int_0^1 d(\mu_1 + \mu_2) \right) \\ &= r \|S''_m\| - \varepsilon(1 - r). \end{aligned}$$

Since u^* and S''_m are bounded in a neighbourhood of δ_0, if $r \to 0$ as $\delta \to \delta_0$, we come to a contradiction. Thus, we can take $r = 1$ and assume that the Lagrange multipliers are locally bounded.

Now let $\delta^1 = (e^1, y^1, d^1)$ and $\delta^2 = (e^2, y^2, d^2)$ be in a neighbourhood of δ_0. Denote by (u^i, α^i, x^i) the corresponding solution and by (v^i, λ^i, μ^i) the Lagrange multipliers. From (6) we have

$$\|u^1\|^2 \leqslant \|u^2\|^2 + \sum_{i=1}^n \lambda_i^1 (y_i^2 - y_i^1) + \int_0^1 (e^1 - e^2)\, d\mu_1^1 + \int_0^1 (d^2 - d^1)\, d\mu_2^1.$$

Furthermore, the optimality yields

$$\begin{aligned} \|u^1\|^2 \geqslant \|u^2\|^2 &+ \sum_{i=1}^n \lambda_i^2 (y_i^2 - y_i^1) + \int_0^1 (e^1 - e^2)\, d\mu_1^2 \\ &+ \int_0^1 (d^2 - d^1)\, d\mu_2^2 + \|u^1 - u^2\|^2. \end{aligned}$$

Combining the last two inequalities we obtain

$$\|u^1 - u^2\|^2 \leqslant \sum_{i=1}^{n} (\lambda_i^1 - \lambda_i^2)(y_i^2 - y_i^1) + \int_0^1 (e^1 - e^2)\, d(\mu_1^1 - \mu_1^2)$$
$$+ \int_0^1 (d^2 - d^1)\, d(\mu_2^1 - \mu_2^2).$$

This inequality and the boundedness of the Lagrange multipliers imply that u is locally 1/2-Lipschitz. By the variation-of-constants formula we have

$$x_2(0) = y_n - y_1 - \int_0^1 (1 - t)\, u(t)\, dt,$$

and hence $x_2(0)$ is locally 1/2-Lipschitz as well. Obviously $x_1(0) = y_1$ is Lipschitz. This completes the proof of the theorem.

6. Remarks

The separation of convex sets is a basic tool for deriving necessary conditions for optimality. The difficulty is that for every concrete problem one has to choose appropriate spaces in order to avoid redundant assumptions. There are numerous duality results in the literature, which, however, are addressed to somewhat different problems that, for example, do not involve constraints corresponding to our interpolation conditions, and the optimal control is assumed to be a bounded measurable or piecewise continuous function, see [1, 5, 6]. Therefore we present here an independent proof of our duality lemma. The proof moreover provides a basis for evaluation of the stability of the solution with respect to perturbations of the data.

If $e = 0$ and $d = +\infty$ then in the lemma $\mu_2 = 0$ and from our theorem we obtain Theorem 3.1 in [8], whose concise proof is based on the Du-Bois–Reymond lemma but cannot be extended, however, to the more general problem we consider here.

The necessary conditions obtained mean that problem (1) can be reduced to a finite-dimensional minimum problem. A straightforward approach to numerical computations is then to treat the new knots as parameters, to construct a feasible cubic spline satisfying the conditions (1)–(3) of the theorem, and then to change appropriately the new knots in order to minimize the norm of the second derivative of the spline. The convergence of this and other algorithms for constrained best interpolation is a subject for future research.

The result presented can be easily extended to the space L_p, $1<p<+\infty$; the cases $p=1$ and $p=+\infty$ need more careful analysis.

As a generalization to (1) one can consider the problem

$$\text{minimize } \|f^{(k)}\| \text{ subject to } f \in \{f \in W^{k,2}: f(t_i)=y_i,\ i=1, ..., n,$$
$$\text{and } e(t) \leqslant f^{(s)}(t) \leqslant d(t) \text{ for all } t \in [0, 1]\},$$

where $k>s\geqslant 0$. (The case $k=s$ is considered, e.g., in [2, 7].) It turns out that for $k-s>3$ the solution of this problem may not be a piecewise polynomial function. Slightly modifying the example studied by Robbins [9] one can construct a monotone interpolation problem (i.e., $s=1$, $e=0$, $d=+\infty$) with $k=4$ for which the first derivative of the solution touches the boundary of the constraint at an infinite sequence of points $t^1, t^2, ...,$ convergent geometrically to a point beyond which $f'=0$. In terms of duality, the measure representing the Lagrange multiplier is concentrated in an infinite sequence of points, that is, the dual problem becomes infinite-dimensional. This type of "landing on" and "taking off" the constraints with infinitely many "jumps" is studied in detail in [1].

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